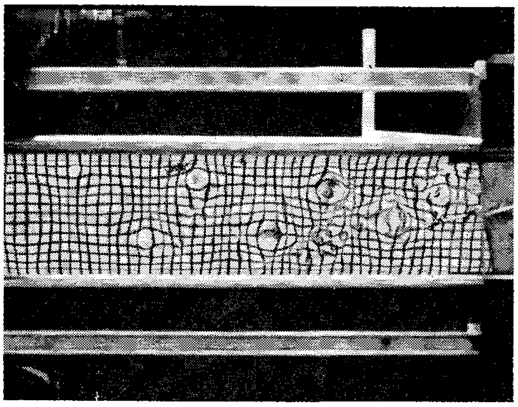

Fig. 5 The vortex street resumes in the magnetic field-free region at the end of the run.

these photographs, Reynolds number (based upon flat-plate width) $\sim 10^4$, magnetic Reynolds number (based upon flat-plate width) $\sim 10^{-3}$, vortex decay time $(4.6\rho/\sigma B_0^2) \sim 1$ sec, and vortex $D_0/H \sim 1$.

Despite the fact that the vortex decay time was long enough to allow observation of the persisting vortices, none can be seen in the magnetic field region of this experiment.

Vortices could only be observed within the magnetic field region when the vortex decay time was increased to ~ 5 sec. In accordance with the notion that the transverse magnetic field suppresses the rotation of existing vortices, the vortices observed within the magnetic field in this case did die out much more rapidly than those found outside the magnetic field.

These observations indicate that the stronger transverse magnetic field did more than merely suppress the vortices created by the flat plate, but that the magnetic field altered the separation characteristics of the boundary layer to prevent the formation of the vortex street. Although a complete solution of the problem is difficult, the Kármán-Pohlhausen technique can be used to determine whether magnetohydrodynamic effects are large enough to possibly influence separation.[1]

Since the still mercury can impress the condition of zero electric field upon the thin boundary layer, the current density at the wall of the moving flat plate must be nearly $\sigma_0 U_0 B_0$, where U_0 is the plate velocity. Consequently, the momentum equation at the plate wall becomes approximately

$$\nu\left(\frac{\partial^2 u}{\partial y^2}\right)_0 = -U_0\frac{dU}{dx} - \frac{\sigma U_0 B_0{}^2}{\rho}$$

and the resulting shape factor Λ' is, therefore, related to the conventional shape factor $\Lambda = (\delta^2/\nu)(dU/dx)$ through the expression

$$\Lambda' = \Lambda\{1 + \sigma B_0{}^2/\rho(dU/dx)\}$$

Note that $\Lambda' \geq \Lambda$ because the induced currents tend to fill out the velocity profile and that separation still occurs when $\Lambda' = -12$. For the experiment just described, $\sigma B_0{}^2/[\rho(dU/dx)] \sim -1$, indicating that the effect of the induced currents in the boundary layer upon separation cannot be neglected for this case. Furthermore, the Hartmann number (based upon flat-plate width) for this experiment was approximately 100, which means that the magnetohydrodynamic forces will tend to decrease the absolute value of Λ' by reducing δ.

The results of these experiments, therefore, appear to be quantitatively in accord with boundary-layer theory, and they demonstrate that magnetic fields can exert a strong influence upon boundary-layer separation. It was also experimentally demonstrated that a magnetic field transverse to the axis of a two-dimensional, wide-angle diffuser could prevent boundary-layer separation.

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Three-Dimensional Symmetric Vortex Flow

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MANY problems of interest in missile or aircraft aerodynamics require a detailed knowledge of the vortex flow due to bodies of revolution or lifting surfaces, and in the following the behavior of symmetrical vortex pattern in the presence of a semicircular section in the crossflow plane has been explored with the help of a simplified model in which the vorticity is moving along the feeding sheets into the cores at all times.

Symmetric vortex separation is exactly identified in the two-dimensional picture of the wake development behind a circular cylinder. The boundary layer separates from the surface of the cylinder at the rearward stagnation point, and the two separation points move symmetrically away from this point around the cylinder. When these boundary-layer separation points reach a certain angular distance from the rearward point, the two regions of vorticity break away from the boundary layer and proceed downstream, forming the wake. Depending upon the Reynolds number, the flow behind the cylinder in the wake is identified as the Stokes flow (without the wake formation), symmetric vortex shedding and anti symmetric vortex formation with rapid transgress into turbulent motion in the wake.

In the three-dimensional flow, the forementioned vortex separation is observed on the leeward side of the slender bodies of revolution at a comparatively large angle of attack in the subsonic to supersonic range. If we imagine a fixed plane in the fluid perpendicular to the axis of the body of revolution, then the body pierces the plane with a velocity $U \sin\alpha$, α being the local body angle of attack and U the velocity. Now, considering the case in which the expanding circle, as the nose pierces the plane, changes slowly with x, x being taken in the direction of the body axis, the flow in the crossflow plane can be looked upon as the two-dimensional one. Such a procedure will not be invalid in view of the conical flow assumption in the slender body theory.

In the present problem of flow past a half cone, the flow in the crossflow plane can be solved by choosing a suitable transformational function that will map the flow plane conformally past the semicircle section into the flow in the crossflow plane past a circular section.[1] Designating the physical plane (i.e., the semicircle plane) by the Z plane and the transformed plane (i.e., the circle plane) by the τ plane, the appropriate transformation mapping the flow conformally between the two planes is given by

$$\left(\frac{Z - 3(3)^{1/2}a/4}{Z + 3(3)^{1/2}a/4}\right) = \left(\frac{\tau - ae^{-i\pi/6}}{\tau + ae^{i\pi/6}}\right)^{3/2} \quad (1)$$

where C, the center of the circle, has been taken as the origin in the τ plane, and infinity has been preserved during the transformation in both planes.[1]

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The flow in the circle plane (i.e., τ plane) has been studied[2] by writing down the complex potential function in the following way:

$$F(\tau) = \phi + i\psi = -iU\alpha(\tau - a^2/\tau) - i\Gamma/2\pi \log[(\tau - \tau_0) \times (\tau + \alpha^2/\tau_0)/(\tau + \bar{\tau}_0)(\tau - \alpha^2/\bar{\tau}_0)] \quad (2)$$

ϕ and ψ being the potential and stream functions of the flow in the crossflow plane, $(\tau_0, -\bar{\tau}_0)$ the coordinates of the centers of the vortex cores, α the local body angle of attack (α being assumed small), and Γ the strength of the vortex.

Then considering the points $T(= ae^{-i'\pi/6})$ and $L(= -ae^{i\pi/6})$,[1] as the stagnation points of the flow, we finally get, after some algebraic simplification, the following relation:

$$\frac{\Gamma/2\pi a U\epsilon_0}{(\alpha/\epsilon_0)} = \frac{(ae^{-i'\pi/6} - \tau_0)(ae^{-i'\pi/6} + \bar{\tau}_0)(ae^{i'\pi/6} - \tau_0)(ae^{i'\pi/6} + \bar{\tau}_0)}{a(\tau_0 + \bar{\tau}_0)(\tau_0\bar{\tau}_0 - a^2)} \quad (3)$$

where ϵ_0 is the semivertical angle of the cone in the circle plane. Another boundary condition is needed for the determination of the flow that, for the present simplified model of the actual flow, is the condition of zero net force on the feeding vortex sheet, because the assumed vortex system (the feeding vortex sheet and the concentrated vortex)[3] must have a zero net force acting on it since only the conical surface and not the fluid can sustain the forces. Application of this idea to the model then requires that the forces on the feeding vortex sheet be canceled by equal and opposite forces on the concentrated vortex, and mathematically this leads to the following relation:

$$-i(\alpha/\epsilon_0)(1 + a^2/\tau_0^2) - i\Gamma/2\pi a U\epsilon_0\{[a\tau_0/(\tau_0^2 + a^2)] - [a/(\tau_0 + \bar{\tau}_0)] - [a\bar{\tau}_0/(\tau_0\bar{\tau}_0 - a^2)]\} = (2\bar{\tau}_0/a - e^{i\pi/6}) \quad (4)$$

Now the relations (3) and (4) will determine τ_0 (i.e., the coordinate of the vortex center) as a function of (α/ϵ_0), (α/ϵ_0) being the dimensionless body angle of attack, and thus the locus of the vortex center, and further, the functional relationship between $\Gamma/2\pi a U\epsilon_0$ and (α/ϵ_0), which are needed for the complete evaluation of the problem. From the knowledge of the flow in the circle plane (i.e., τ plane), the flow in the physical plane (i.e., Z plane) can be determined by the help of the transformation given in Eq. (1).

The lift function in the present case can be calculated with the help of the momentum consideration,[3,4] i.e.,

$$L = -\rho U \, Re\left[\int_{C_1} F(Z)\, dZ\right] \quad (5)$$

where Re stands for the real part of the expression under the sign of integration and C_1 is the contour just containing the wake, or

$$L = -\rho U \, Re\left[\int_C F(\tau)\frac{dZ}{d\tau}\, d\tau\right] \quad (6)$$

where the integration is being carried out at a large distance from the wake in the circle plane; by the usual complex integration technique, it follows that

$$C_L/\epsilon_1^2 = 6.213(\alpha/\epsilon_1) + 4.734(\alpha/\epsilon_1)F(\alpha/\epsilon_1)$$

where C_L is the lift coefficient defined by the relation $C_L = L/(\rho U^2 S/2)$, S being the area of the curved surface of the semicone, ϵ_1 the semiaperture of the half-cone in the physical plane ($\epsilon_1 \cong 1.30\epsilon_0$ approximately), and

$$F\left(\frac{\alpha}{\epsilon_1}\right) = \frac{(ae^{-i\pi/6} - \tau_0)(ae^{-i\pi/6} + \bar{\tau}_0)(ae^{i\pi/6} - \tau_0)(ae^{i\pi/6} + \bar{\tau}_0)}{a^2\tau_0\bar{\tau}_0}$$

to which the right-hand expression can be reduced.

Since (α/ϵ_1) is a function of the space coordinates, $F(\alpha/\epsilon_1)$ is assumed to be analytic in the ring region bounded by the two concentric circles (the inner one being the circle of radius a forming the conical surface in the τ plane and the outer one being a large circle in the wake in the τ plane), and it may be expressed as

$$F(\alpha/\epsilon_1) = [(A_{-1})/(\alpha/\epsilon_1) + A_0 + A_1(\alpha/\epsilon_1) + \ldots]$$

[an expansion similar to Laurent's series stopping at the left at $1/(\alpha/\epsilon_1)$, since the lift coefficient can not be infinite as $(\alpha/\epsilon_1) \to 0$] so that

$$C_L/\epsilon_1^2 = 4.734\, A_{-1} + (\alpha/\epsilon_1)[6.213 + 4.7A_0 \ldots] \quad (7)$$

so that when $(\alpha/\epsilon_1) \to 0$, (C_L/ϵ_1^2) becomes equal to $4.734\, A_{-1}$ where A_{-1} may be either positive or negative. However, one may attempt to evaluate the case of $(\alpha/\epsilon_1) \to 0$, i.e., for zero angle of attack, directly.

The equation to the conical surface (namely the half-cone)[4] can be written as

$$Z_+ \equiv f(x, y) = (K^2x^2 - y^2)^{1/2} \quad (8)$$

where $K = \tan\epsilon_1$, ϵ_1 being the semivertical angle of the semicone, and the orientation of the coordinates is the following: the x axis is parallel to the direction of the main flow, the y axis points toward the starboard, and the z axis points upward.

The proper boundary condition (of the three-dimensional flow) to be satisfied at any point on the conical surface is given by[5]

$$w - U(\partial f/\partial x) = 0$$

i.e.,

$$w/U = (\partial f/\partial x) = K^2x/(K^2x^2 - y^2)^{1/2} \quad (9)$$

where w is the induced velocity of the flow in the Z direction, and U is the mainstream flow in the x direction.

If Γ is the circulation around any section of the half-cone, the normal induced velocity at any point y_1 of the span is determined by the relation

$$w(y_1) = \frac{1}{4\pi}\int_{-s}^{+s}\frac{d\Gamma/dy}{y_1 - y}\, dy$$

where $s = Kx$ in the present case.[1] Putting $y = -s\cos\theta$ where θ varies from 0 to π across the span of the semicircle, the circulation Γ may be expressed in a Fourier's series:

$$\Gamma = 4sU\sum_1^{\infty} A_n \sin n\theta$$

the values of the coefficients A_n must be determined in accordance with the two equations connecting Γ and w. The normal induced velocity at the point y_1 or θ_1 of the semicircle becomes

$$w(\theta_1) = \frac{U}{\pi}\int_0^{\pi}\frac{\Sigma n A_n \cos n\theta}{(\cos\theta - \cos\theta_1)}\, d\theta = U\sum_1^{\infty} nA_n\frac{\sin n\theta_1}{\sin\theta_1}$$

and therefore,

$$\frac{w(\theta)}{U} = \sum_1^{\infty} uA_n\left(\frac{\sin n\theta}{\sin\theta}\right) = (\partial f/\partial x) \text{ [from condition (9)]}$$

from which we find that

$$A_n(1 - \cos n\pi) = \Gamma\pi/2$$

The lift function

$$L = \int_{-s}^{+s}\rho U\Gamma\, dy$$

and therefore, the lift coefficient is given by

$$C_L/\epsilon_1^2 = (\pi/0.845)(a_1/a)^2$$

where $a_1 = Kx = s$; in the present case, it is the radius of the semicircle in the physical plane, and a is the radius in the transformed plane.

Remarks

It may be remarked here in this connection that, in the Brown and Michael work,[3] the condition of zero net force configuration has been satisfied in the physical plane (i.e., the delta-wing plane). The transformed plane, i.e., θ plane, has been brought in the picture to facilitate the formulation of the potential flow corresponding to the physical plane. However, in the present problem, the flow investigation has been carried out in the transformed plane because the condition of zero net force configuration of the vortex sheet and the concentrated vortex can be satisfied in the transformed plane, since, in this plane also, the flow separates from the surface, and the vortex sheet is formed from the feeding points on either side of the body. It is true that, depending upon the geometrical shape of the body in the crossflow plane, the geometrical shape of the locus of the vortex center will also change; this is being reflected in some of the analytical derivations in the transformed as well as in the physical plane. One can easily write down the analytical relations in the physical plane in the following way. In place of Eq. (1), we start with the equation

$$\left(\frac{\tau - ae^{-i\pi/6}}{\tau + ae^{2'\pi/6}}\right) = \left(\frac{Z - [3(3)^{1/2}/4]a}{Z + [3(3)^{1/2}/2]a}\right)^{2/3} \qquad (10)$$

from which we put

$$\tau/a = f(Z/a) \qquad (11)$$

where

$$f\left(\frac{Z}{a}\right) = \frac{(3)^{1/2}}{2} \times \frac{\{1 + [3(3)^{1/2}/4](a/Z)\}^{2/3} + \{1 - [3(3)^{1/2}/4](a/Z)\}^{2/3}}{\{1 + [3(3)^{1/2}/4](a/Z)\}^{2/3} - \{1 - [3(3)^{1/2}/4](a/Z)\}^{2/3}} \qquad (12)$$

The equation corresponding to Eq. (2) can be written as

$$F(Z/a) = -iU\alpha a(f - 1/f) - i\Gamma/2\pi \log(f - f_0) - i\Gamma/2\lambda \log(f + 1/f_0) + i\Gamma/2\lambda \log(f + \bar{f}_0) + i\Gamma/2\lambda \log(f - 1/\bar{f}_0) \qquad (13)$$

The lift function can be written down in the following way:

$L(x)$ = the sectional lift in the crossflow plane

$$= -\rho U \, Re \left[\int_C F(Z)\, dZ\right] = 21 \; \pi\rho U^2 a^2 \alpha + \rho U \Gamma a \frac{(f_0 + \bar{f}_0)(f_0\bar{f}_0 - 1)}{f_0\bar{f}_0}$$

or, expressing the lift coefficient C_L as defined earlier, we find that

$$C_L = 6.23\left(\frac{\alpha}{\epsilon_1}\right) + 4.741 \frac{\Gamma}{2\pi a U \epsilon_1} \frac{(f_0 + \bar{f}_0)(f_0\bar{f}_0 - 1)}{f_0\bar{f}_0}$$

However, the stagnation-point consideration in the physical plane (cf., Eq. 3) leads to the following equation:

$$\frac{\Gamma/2\pi a U\epsilon_1}{(\alpha/\epsilon_1)} = \frac{(f_1 - f_0)(f_1 + \bar{f}_0)(f_0 + \bar{f}_1)(\bar{f}_1 - \bar{f}_0)}{(f_0 + \bar{f}_0)(f_0\bar{f}_0 - 1)} \qquad (14)$$

and therefore,

$$\frac{C_L}{\epsilon_1^2} = 6.23\left(\frac{\alpha}{\epsilon_1}\right) + 4.741\left(\frac{\alpha}{\epsilon_1}\right) \times \frac{(f_1 - f_0)(f_1 + \bar{f}_0)(f_0 + \bar{f}_1)(\bar{f}_1 - \bar{f}_0)}{f_0\bar{f}_0} \qquad (15)$$

The condition of zero net force configuration in the physical plane now becomes

$$\left[-iU\alpha a\left(1 + \frac{1}{f_0^2}\right) - \frac{i\Gamma}{2\pi}\frac{1}{[f_0 + (1/f_0)]} + \frac{i\Gamma}{2\pi}\frac{1}{f_0 - (1/\bar{f}_0)} + \frac{i\Gamma}{2\pi}\frac{1}{f_0 + \bar{f}_0}\right] - \frac{i\Gamma}{4\pi}\frac{f_0''}{(f_0')^2} = U\epsilon_1\left(2\frac{Z_0}{a_1} - 1\right)\frac{1}{(df/dZ)_0} \qquad (16)$$

In case $f''(Z/a)$ can be neglected (which is possible only when the vortex center is very close to the feeding point) Eq. (16) becomes equivalent to Eq. (4) if we can interpret $(df/dZ)_0$ as the magnification of the scale of transformation between the physical plane and the transformed plane.

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Torque on a Satellite Due to Gravity Gradient and Centrifugal Force

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In order to analyze the rotational dynamics of earth-orbiting satellites during the preliminary design of their attitude control systems, it is customary and usually necessary to determine the approximate values of gravity-gradient torques on the satellites. Very often these values are based strictly on the gradient in gravitational force across the distributed mass of the satellite. For the case of a symmetric satellite in a circular orbit, such values might typically be calculated using the expression given by Nidey.[1] When this expression [Eq. (18) of Ref. 1] is written in scalar form, the result is

$$T_g = (\tfrac{3}{2})\omega_0^2(I_s - I_t)\sin 2\beta' \qquad (1)$$

where

- T_g = magnitude of the instantaneous gravity-gradient torque vector, ft-lb
- ω_0 = orbital revolution rate, rad/sec
- I_s = moment of inertia about the axis of symmetry, slugs-ft²
- I_t = transverse moment of inertia, slugs-ft²
- β' = angle from the symmetry axis to local vertical (when $I_s < I_t$) or the horizontal plane (when $I_s > I_t$), rad

The torque acts about the transverse body axis that lies instantaneously in the horizontal plane, and the torque direction is such as to decrease β'. There are several examples in the literature wherein expressions equivalent to Eq. (1) are used to derive values of gravity-gradient torques.[2–5] These results imply erroneously that their results give a

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